

Relative Nash Welfarism

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Abstract

Relative Nash welfarism is a solution to the problem of aggregating von Neumann-Morgenstern preferences over a set of lotteries. It ranks such lotteries according to the product of any collection of 0-normalized von Neumann-Morgenstern utilities they generate. We show that this criterion is characterized by the Weak Pareto Principle, Anonymity, and Independence of Harmless Expansions: the social ranking of two lotteries is unaffected by the addition of any alternative that every agent deems at least as good as the one she originally found worst. Relative Nash welfarism is more appealing than relative utilitarianism in contexts where the *best* relevant alternative for an agent is difficult to identify with confidence.

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1. Introduction

The most popular solution to the problem of aggregating von Neumann-Morgenstern (hereafter, vN-M) preference orderings over a set of lotteries is *relative utilitarianism* : Dhillon (1998), Karni (1998), Dhillon and Mertens (1999), Segal (2000), Börgers and Choo (2017). It consists in ranking such lotteries according to the sum of the (0,1)-normalized von Neumann-Morgenstern utilities they generate. Contrary to Harsanyi's (1955) classical utilitarianism, relative utilitarianism does not require an a priori knowledge of individual utilities and therefore constitutes a bona fide ordinal aggregation rule.

Of course, the recommended social ranking of two lotteries depends upon the set of (pure) alternatives that are considered relevant. In particular, it may be affected by the addition of an alternative that an individual deems worse than the one she initially found worst, or better than the one she initially found best. In many applications, identifying

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the worst relevant alternative for each individual may be relatively easy, but determining the best one is difficult.

As an illustration, consider the problem of developing medical treatment against two diseases, A and B . Let x_d denote the quality of the treatment developed against disease d : say that $x_d = 0$ if no treatment exists, $x_d = \frac{1}{2}$ if a good treatment is made available, and $x_d = 1$ if the treatment is excellent (these numbers are a convenient way of indexing the possibilities but have no meaning –we could use $x_d = \alpha, \beta, \gamma$ instead). The relevant alternatives are all the pairs $x = (x_A, x_B)$ in the set

$$X = \left\{0, \frac{1}{2}, 1\right\} \times \left\{0, \frac{1}{2}, 1\right\}.$$

Agent 1 suffers from disease A ; her preferences over the lotteries on X are represented by the vN-M utility function $u_1(x) = x_A$ on X . Agent 2 suffers from disease B and her preferences admit the vN-M representation $u_2(x) = x_B$. Observe that, given X , the functions u_1, u_2 are $(0, 1)$ -normalized: $\min_X u_i = 0$ and $\max_X u_i = 1$ for $i = 1, 2$. Relative utilitarianism deems the alternatives $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ equally good because both generate a sum of $(0, 1)$ -normalized utilities equal to $\frac{1}{2}$.

Suppose now that, in fact, an excellent treatment cannot possibly be developed against B . The set of relevant alternatives then becomes

$$Y = \left\{0, \frac{1}{2}, 1\right\} \times \left\{0, \frac{1}{2}\right\}.$$

Given Y , the $(0, 1)$ -normalized vN-M representations of the preferences are now $v_1(x) = u_1(x) = x_A$ and $v_2(x) = 2u_2(x) = 2x_B$. Relative utilitarianism deems $(0, \frac{1}{2})$ preferable to $(\frac{1}{2}, 0)$.

Thus, in order to decide whether a good treatment against A (and no treatment against B) is preferable to a good treatment against B (and no treatment against A), society needs to know whether an excellent treatment against B (and A) is possible or not. There need not be anything morally wrong with this view, but it may be difficult to implement in practice.

In contexts where the best relevant alternative for each individual is hard to determine, we submit that *relative Nash welfarism* is a more appealing criterion. It works as follows: choose for each possible individual preference ordering an arbitrary 0-normalized von Neumann-Morgenstern representation; at any preference profile, rank lotteries according to the product of the 0-normalized utilities they generate. The choice of the particular 0-normalizations is irrelevant because all 0-normalized von Neumann-Morgenstern numerical representations of a given preference ordering are positive multiples of each other.

In the example above, u_1, u_2 are 0-normalized for both X and Y : $\min_X u_i = \min_Y u_i = 0$ for $i = 1, 2$. Relative Nash welfarism deems the alternatives $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ equally good independently of whether the set of relevant alternatives is X or Y . Of course, the social preference still requires a correct specification of the worst relevant alternative for each individual. In many cases, this may not be an impossible task.

This example illustrates a general property of the relative Nash ranking: it is unaffected by the addition of any alternative that all individuals find at least as good as the one they initially found worst – *even if such a new alternative is better than the one they initially found best*. We show that relative Nash welfarism is the only weakly Paretian and anonymous criterion satisfying this “Independence of Harmless Alternatives” property.

Our theorem is a variant of the result offered by Kaneko and Nakamura (1979). There are three differences. The first and most important one is conceptual: Kaneko and Nakamura’s analysis is restricted to problems where all agents agree on what the worst relevant alternative is: a lottery is then evaluated according to the product of the vN-M utility gains it generates with respect to this *common* worst alternative. The scope of applicability of relative Nash welfarism, as we define it, is much broader: just like relative utilitarianism, it allows society to rank the set of lotteries generated by a finite collection of alternatives for *any* profile of vN-M preferences. When the agents disagree on which alternative is worst, the vN-M utility gain of each agent is measured with respect to the alternative that *she* finds worst, and a lottery is evaluated according to the product of the vN-M utility gains computed in this way.

The remaining two differences are technical, though important. Kaneko and Nakamura’s independence axiom embodies an assumption of neutrality which we dispense with, and a continuity axiom which we also do not require. Note that Kaneko (1984) also dispenses with continuity, albeit in a framework with a continuum of individuals.

2. Framework

Let A be an infinite set of *conceivable (social) alternatives* and let \mathcal{A} be the set of finite subsets of A containing at least two elements. For each $X \in \mathcal{A}$, let $\Delta(X) = \left\{ a \in [0, 1]^X \mid \sum_{x \in X} a(x) = 1 \right\}$ be the set of lotteries on X . The elements of X are the *relevant alternatives* and the elements of $\Delta(X)$ are the *relevant lotteries*. If $x \in X$, we abuse notation and also use x to denote the lottery in $\Delta(X)$ assigning probability 1 to the alternative x .

For any $X \in \mathcal{A}$, a *preference ordering* over $\Delta(X)$ is an ordering $R \subseteq \Delta(X) \times \Delta(X)$. We call R *non-degenerate* if $R \neq \Delta(X) \times \Delta(X)$. A *vN-M representation* of R is a function $u : \Delta(X) \rightarrow \mathbb{R}$ such (i) $u(a) \geq u(b) \Leftrightarrow aRb$ for all $a, b \in \Delta(X)$ and (ii) $u(\lambda a + (1 - \lambda)b) = \lambda u(a) + (1 - \lambda)u(b)$ for all $a, b \in \Delta(X)$ and all $\lambda \in [0, 1]$. If such a function exists, R is a *vN-M preference ordering*. We call a function u satisfying condition (ii) a *vNM function*. Let $\mathcal{R}(X)$ denote the set of all preference orderings over $\Delta(X)$ and let $\mathcal{R}^*(X)$ denote the subset of non-degenerate vN-M preference orderings. Write $\mathcal{R} = \cup_{X \in \mathcal{A}} \mathcal{R}(X)$ and $\mathcal{R}^* = \cup_{X \in \mathcal{A}} \mathcal{R}^*(X)$.

Let $N = \{1, \dots, n\}$ be a fixed finite set of individuals. A *(social choice) problem* is a list (X, R_N) where $X \in \mathcal{A}$ and $R_N = (R_1, \dots, R_n) \in \mathcal{R}^*(X)^N$. We simply call R_N a *preference profile* –but keep in mind that R_1, \dots, R_n are non-degenerate vN-M preference orderings. The set of all problems is denoted by \mathcal{P} . An *(aggregation) rule* is a mapping $\mathbf{R} : \mathcal{P} \rightarrow \mathcal{R}$ such that $\mathbf{R}(X, R_N) \in \mathcal{R}(X)$ for every $(X, R_N) \in \mathcal{P}$. Note that the social preference

ordering $\mathbf{R}(X, R_N)$ need not be a vN-M preference ordering and may be degenerate.

3. Theorem

For any $i \in N$, $X \in \mathcal{A}$, and $R_i \in \mathcal{R}^*(X)$, denote by $A_0(X, R_i)$ and $A_1(X, R_i)$ the sets of worst and best lotteries in $\Delta(X)$ according to R_i . A vN-M representation u_i of R_i is *0-normalized* if $u_i(a) = 0$ for all $a \in A_0(X, R_i)$. It is $(0, 1)$ -normalized if, in addition, $u_i(a) = 1$ for all $a \in A_1(X, R_i)$. We denote by $\mathcal{U}_0(X, R_i)$ the set of 0-normalized vN-M representations of R_i : observe that if $u_i \in \mathcal{U}_0(X, R_i)$, then $v_i \in \mathcal{U}_0(X, R_i)$ if and only if $v_i = \lambda u_i$ for some positive real number λ . The $(0, 1)$ -normalized vN-M representation of R_i is unique: we denote it $u^*(\cdot, X, R_i)$.

The *relative Nash aggregation rule* \mathbf{R}^* is defined as follows: for all $(X, R_N) \in \mathcal{P}$ and all $a, b \in \Delta(X)$,

$$a\mathbf{R}^*(X, R_N)b \Leftrightarrow \prod_{i \in N} u_i(a) \geq \prod_{i \in N} u_i(b) \text{ for all } (u_1, \dots, u_n) \in \prod_{i \in N} \mathcal{U}_0(X, R_i). \quad (3.1)$$

Since every 0-normalized vN-M representation of R_i is a positive linear transformation of $u^*(\cdot, X, R_i)$, (3.1) is equivalent to

$$a\mathbf{R}^*(X, R_N)b \Leftrightarrow \prod_{i \in N} u^*(a, X, R_i) \geq \prod_{i \in N} u^*(b, X, R_i).$$

The following notation and terminology will be needed to state our axiomatic characterization of the relative Nash aggregation rule. The symbols P_i and I_i denote the strict preference and indifference relations associated with the individual preference ordering R_i , and $\mathbf{P}(X, R_N)$ and $\mathbf{I}(X, R_N)$ are the strict social preference and indifference relations associated with $\mathbf{R}(X, R_N)$. We denote by $\Pi(N)$ the set of permutations on N . If R_N is a preference profile on $\Delta(X)$ and $\sigma \in \Pi(N)$, then $\sigma R_N = (\sigma R_1, \dots, \sigma R_n)$ is the profile on $\Delta(X)$ given by $\sigma R_{\sigma(i)} = R_i$ for all $i \in N$. Finally, if $(X, R_N), (X', R'_N) \in \mathcal{P}$ and $X \subseteq X'$, we say that R'_N *coincides* with R_N on $\Delta(X)$ if $R'_i \cap (\Delta(X) \times \Delta(X)) = R_i$ for all $i \in N$. Similarly, $\mathbf{R}(X', R'_N)$ *coincides* with $\mathbf{R}(X, R_N)$ on $\Delta(X)$ if $\mathbf{R}(X', R'_N) \cap (\Delta(X) \times \Delta(X)) = \mathbf{R}(X, R_N)$.

The conditions we impose on the rule \mathbf{R} are the following.

Weak Pareto Principle. For all $(X, R_N) \in \mathcal{P}$ and $a, b \in \Delta(X)$, (i) if aP_ib for all $i \in N$, then $a\mathbf{P}(X, R_N)b$, and (ii) if aI_ib for all $i \in N$, then $a\mathbf{I}(X, R_N)b$.

We refer to part (ii) of this condition as *Pareto Indifference*.

Anonymity. For all $(X, R_N) \in \mathcal{P}$ and $\sigma \in \Pi(N)$, $\mathbf{R}(X, R_N) = \mathbf{R}(X, \sigma R_N)$.

Independence of Harmless Expansions. For all $(X, R_N), (X', R'_N) \in \mathcal{P}$, if $X \subseteq X'$, R'_N coincides with R_N on $\Delta(X)$, and $a'R'_i a_i$ for all $a' \in X'$, all $a_i \in A_0(X, R_i)$, and all $i \in N$, then $\mathbf{R}(X', R'_N)$ coincides with $\mathbf{R}(X, R_N)$ on $\Delta(X)$.

These three conditions characterize relative Nash welfarism:

Theorem. *The aggregation rule \mathbf{R} satisfies the Weak Pareto Principle, Anonymity, and Independence of Harmless Expansions if and only if $\mathbf{R} = \mathbf{R}^*$.*

4. Proof

We begin with a lemma showing that Pareto Indifference and Independence of Harmless Expansions implies a strong form of neutrality. Let $\Pi(A)$ denote the set of permutations on A . If $(X, R_N) \in \mathcal{P}$, $\pi \in \Pi(A)$, $a \in \Delta(X)$, and $R_N \in \mathcal{R}^*(X)^N$, denote by a^π the lottery on $\pi(X)$ given by $a^\pi(\pi(x)) = a(x)$ for all $x \in X$, and denote by R_N^π the preference profile on $\pi(X)$ given by $a^\pi R_i^\pi b^\pi \Leftrightarrow a R_i b$ for all $i \in N$ and all $a, b \in \Delta(X)$.

Neutrality. For all $(X, R_N) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(A)$, $a\mathbf{R}(X, R_N)b \Leftrightarrow a^\pi\mathbf{R}(\pi(X), R_N^\pi)b^\pi$.

Lemma. *If the aggregation rule \mathbf{R} satisfies Pareto Indifference and Independence of Harmless Expansions, then \mathbf{R} satisfies Neutrality.*

This result is a corollary to Lemma 1 in Sprumont (2013); we include a full proof here to make the presentation self-contained.

Proof. Let \mathbf{R} satisfy Pareto Indifference and Independence of Harmless Expansions. Let $(X, R_N) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(A)$. We prove that $a\mathbf{R}(X, R_N)b \Rightarrow a^\pi\mathbf{R}(\pi(X), R_N^\pi)b^\pi$. The converse implication follows immediately since $a = (a^\pi)^{\pi^{-1}}$, $b = (b^\pi)^{\pi^{-1}}$, $X = \pi^{-1}(\pi(X))$, and $R_N = (R_N^\pi)^{\pi^{-1}}$. Let us thus assume that

$$a\mathbf{R}(X, R_N)b. \quad (4.1)$$

Step 1. $a^\pi\mathbf{R}(\pi(X), R_N^\pi)b^\pi$ if $\pi(X) \cap X = \emptyset$.

Let $\bar{X} = X \cup \pi(X)$. For each $i \in N$, let \bar{R}_i be the vN-M preference ordering over $\Delta(\bar{X})$ which coincides with R_i on $\Delta(X)$ and is such that $x\bar{R}_i\pi(x)$ for all $x \in X$. This is well defined because $\pi(X) \cap X = \emptyset$. Observe that \bar{R}_i coincides with R_i^π on $\Delta(\pi(X))$. Moreover, $x\bar{R}_ia$ for all $x \in \bar{X}$ and all $a \in A_0(X, R_i) \cup A_0(\pi(X), R_i^\pi)$. Let $\bar{R}_N = (\bar{R}_1, \dots, \bar{R}_n)$. Applying Independence of Harmless Expansions to (4.1),

$$a\mathbf{R}(\bar{X}, \bar{R}_N)b. \quad (4.2)$$

Since $a^\pi\bar{I}_ia$ and $b^\pi\bar{I}_ib$ for all $i \in N$, Pareto Indifference implies $a^\pi\mathbf{I}(\bar{X}, \bar{R}_N)a$ and $b^\pi\mathbf{I}(\bar{X}, \bar{R}_N)b$. Hence from (4.2),

$$a^\pi\mathbf{R}(\bar{X}, \bar{R}_N)b^\pi. \quad (4.3)$$

Applying Independence of Harmless Expansions to (4.3) and recalling that \bar{R}_N coincides with R_N^π on $\Delta(\pi(X))$, we obtain $a^\pi\mathbf{R}(\pi(X), R_N^\pi)b^\pi$.

Step 2. $a^\pi\mathbf{R}(\pi(X), R_N^\pi)b^\pi$.

Choose $\rho \in \Pi(A)$ such that $\rho(X) \cap X = \rho(X) \cap \pi(X) = \emptyset$. By Step 1, (4.1) implies

$$a^\rho\mathbf{R}(\rho(X), R_N^\rho)b^\rho. \quad (4.4)$$

Next consider the permutation $\pi \circ \rho^{-1} \in \Pi(A)$. Since $(\pi \circ \rho^{-1})(\rho(X)) \cap \rho(X) = \emptyset$, Step 1 and (4.4) imply

$$(a^\rho)^{\pi \circ \rho^{-1}}\mathbf{R}((\pi \circ \rho^{-1})(\rho(X)), (R_N^\rho)^{\pi \circ \rho^{-1}})(b^\rho)^{\pi \circ \rho^{-1}}. \quad (4.5)$$

By definition, $(\pi \circ \rho^{-1})(\rho(X)) = \pi(X)$. Moreover, $(a^\rho)^{\pi \circ \rho^{-1}} = a^\pi$ since $(a^\rho)^{\pi \circ \rho^{-1}}(\pi(x)) = (a^\rho)^{\pi \circ \rho^{-1}}((\pi \circ \rho^{-1})(\rho(x))) = a^\rho(\rho(x)) = a(x)$ for all $x \in X$. Likewise, $(b^\rho)^{\pi \circ \rho^{-1}} = b^\pi$ and $(R_N^\rho)^{\pi \circ \rho^{-1}} = R_N^\pi$. Hence (4.5) reduces to $a^\pi \mathbf{R}(\pi(X), R_N^\pi) b^\pi$. ■

Proof of the Theorem. The proof of the “if” statement is straightforward. To prove the converse statement, fix an aggregation rule \mathbf{R} satisfying the Weak Pareto Principle, Anonymity, and Independence of Harmless Expansions. This rule satisfies Pareto Indifference, hence also Neutrality, by the above lemma.

If $(X, R_N) \in \mathcal{P}$ and $u_i \in \mathcal{U}_0(X, R_i)$ for each $i \in N$, define $u_N : \Delta(X) \rightarrow \mathbb{R}_+^N$ by $u_N(a) = (u_1(a), \dots, u_n(a))$ for all $a \in \Delta(X)$. With some abuse of notation, let $\mathcal{U}_0(X, R_N) = \prod_{i \in N} \mathcal{U}_0(X, R_i)$. Define the binary relations \succ , \sim , and \succsim on \mathbb{R}_+^N as follows:

- (i) $v \succ w$ if and only if there exist $(X, R_N) \in \mathcal{P}$, $u_N \in \mathcal{U}_0(X, R_N)$, and $a, b \in X$ such that $u_N(a) = v$, $u_N(b) = w$, and $a \mathbf{P}(X, R_N) b$,
- (ii) $v \sim w$ if and only if there exist $(X, R_N) \in \mathcal{P}$, $u_N \in \mathcal{U}_0(X, R_N)$, and $a, b \in X$ such that $u_N(a) = v$, $u_N(b) = w$, and $a \mathbf{I}(X, R_N) b$,
- (iii) $v \succsim w$ if and only if $v \succ w$ or $v \sim w$.

Step 1. \succsim is an ordering.

To prove reflexivity and completeness of \succsim , fix two (possibly equal) vectors $v, w \in \mathbb{R}_+^N$. Let $a, b, c, d \in A$ be four distinct alternatives and let $X = \{a, b, c, d\}$. For each $i \in N$, choose a number $z_i \in \mathbb{R}_+$ such that $z_i \neq v_i, w_i$, and let $u_i : \Delta(X) \rightarrow \mathbb{R}_+$ be the (unique) vN-M function such that $u_i(a) = v_i$, $u_i(b) = w_i$, $u_i(c) = 0$, and $u_i(d) = z_i$. Let R_i be the preference ordering on $\Delta(X)$ represented by u_i : by construction, $R_i \in \mathcal{R}^*(X)$. Letting $u_N := (u_1, \dots, u_n)$ and $R_N = (R_1, \dots, R_N)$, we have $(X, R_N) \in \mathcal{P}$ and $u_N \in \mathcal{U}_0(X, R_N)$. Since $\mathbf{R}(X, R_N)$ is complete and reflexive, $a \mathbf{R}(X, R_N) b$ or $b \mathbf{R}(X, R_N) a$, implying that $v \succsim w$ or $w \succsim v$.

To prove transitivity of \succsim , fix $v^1, v^2, v^3 \in \mathbb{R}_+^N$ such that $v^1 \succsim v^2 \succsim v^3$. By definition, there exist $(X^1, R_N^1), (X^2, R_N^2) \in \mathcal{P}$, $u_N^1 \in \mathcal{U}_0(X^1, R_N^1)$, $u_N^2 \in \mathcal{U}_0(X^2, R_N^2)$, $a^1, b^1 \in X^1$, and $a^2, b^2 \in X^2$ such that

$$u_N^1(a^1) = v^1, \quad u_N^1(b^1) = v^2 = u_N^2(a^2), \quad \text{and} \quad u_N^2(b^2) = v^3, \quad (4.6)$$

and

$$a^1 \mathbf{R}(X^1, R_N^1) b^1 \quad \text{and} \quad a^2 \mathbf{R}(X^2, R_N^2) b^2. \quad (4.7)$$

By Neutrality, we may assume that $X^1 \cap X^2 = \emptyset$. Let $X = X^1 \cup X^2$. For each $i \in N$, let $u_i : \Delta(X) \rightarrow \mathbb{R}_+$ be the vN-M function such that

$$u_i(x) = \begin{cases} u_i^1(x) & \text{if } x \in X^1, \\ u_i^2(x) & \text{if } x \in X^2. \end{cases} \quad (4.8)$$

Let R_i be the vN-M preference ordering on $\Delta(X)$ represented by u_i , let $u_N = (u_1, \dots, u_n)$, and let $R_N = (R_1, \dots, R_n)$.

Note that R_N coincides with R_N^1 on $\Delta(X^1)$ and with R_N^2 on $\Delta(X^2)$. Moreover, because $u_N^1 \in \mathcal{U}_0(X^1, R_N^1)$ and $u_N^2 \in \mathcal{U}_0(X^2, R_N^2)$, (4.8) implies that $xR_i a_i$ for all $x \in X$, all $a_i \in A_0(X^1, R_i^1) \cup A_0(X^2, R_i^2)$, and all $i \in N$. We may therefore apply Independence of Harmless Expansions to (4.7) and conclude

$$a^1 \mathbf{R}(X, R_N) b^1 \text{ and } a^2 \mathbf{R}(X, R_N) b^2.$$

On the other hand, (4.6) and (4.8) imply $b^1 I_i a^2$ for all $i \in N$, hence by Pareto Indifference,

$$b^1 \mathbf{I}(X, R_N) a^2.$$

Transitivity of $\mathbf{R}(X, R_N)$ now implies $a^1 \mathbf{R}(X, R_N) b^2$. Since $(X, R_N) \in \mathcal{P}$, $u_N \in \mathcal{U}_0(X, R_N)$, and $u_N(a^1) = v^1$ and $u_N(b^2) = v^3$, the definition of \succsim gives us $v^1 \succsim v^3$.

Step 2. $v \succsim w \Leftrightarrow \prod_{i \in N} v_i \geq \prod_{i \in N} w_i$ for all $v, w \in \mathbb{R}_+^N$.

We use the notation $\geq, >, \gg$ to write inequalities in \mathbb{R}_+^N . Because \mathbf{R} satisfies the Weak Pareto Principle, the ordering \succsim is *weakly monotonic*: $v \gg w \Rightarrow v \succ w$. Because \mathbf{R} satisfies Anonymity, \sim is *symmetric*: $v \sim \sigma v$ for all $\sigma \in \Pi(N)$, where σv is the vector defined by $(\sigma v)_{\sigma(i)} = v_i$ for all $i \in N$. Finally, because \mathbf{R} satisfies Independence of Harmless Alternatives, \succsim is *scale invariant*: $v \succsim w \Leftrightarrow \lambda * v \succsim \lambda * w$ for all $\lambda \in \mathbb{R}_{++}^N$, where $\lambda * v = (\lambda_1 v_1, \dots, \lambda_n v_n)$. We omit the straightforward proofs of these three facts.

Step 2.1. $v \sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i)$ for all $v \in \mathbb{R}_+^N$.

For $k = 0, \dots, n$, define $V^k = \{v \in \mathbb{R}_+^N \mid \{i \in N \mid v_i \neq 1\} \leq k\}$, the set of vectors in \mathbb{R}_+^N having at most k coordinates different from 1. Note that $V^0 = \{(1, \dots, 1)\}$ and $V^n = \mathbb{R}_+^N$. Trivially,

$$v \sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i) \text{ for all } v \in V^0.$$

Next, we proceed by induction: we fix k such that $0 \leq k < n$, make the induction hypothesis

$$v \sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i) \text{ for all } v \in V^k.$$

and prove that $v \sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i)$ for all $v \in V^{k+1}$.

Let $v \in V^{k+1}$ and, to avoid triviality, suppose $v \notin V^k$: exactly $k+1$ coordinates of v differ from 1. Without loss of generality, say $v = (v_1, \dots, v_{k+1}, \underbrace{1, \dots, 1}_{n-k-1})$, with $v_i \neq 1$ for $i = 1, \dots, k+1$. By symmetry of \sim ,

$$(v_1, \dots, v_k, \underbrace{1, \dots, 1}_{n-k}) \sim (v_1, \dots, v_{k-1}, 1, v_k, \underbrace{1, \dots, 1}_{n-k-1}). \quad (4.9)$$

Since \succsim is scale invariant, (4.9) implies

$$v \sim (v_1, \dots, v_{k-1}, 1, v_k \cdot v_{k+1}, \underbrace{1, \dots, 1}_{n-k-1}).$$

By the induction hypothesis,

$$(v_1, \dots, v_{k-1}, 1, v_k \cdot v_{k+1}, \underbrace{1, \dots, 1}_{n-k-1}) \sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i),$$

hence by transitivity of \sim ,

$$v \sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i).$$

Step 2.2. $v \succsim w \Leftrightarrow \prod_{i \in N} v_i \geq \prod_{i \in N} w_i$ for all $v, w \in \mathbb{R}_+^N$.

First, observe that

$$v \sim (\prod_{i \in N} v_i^{\frac{1}{n}}, \dots, \prod_{i \in N} v_i^{\frac{1}{n}}) \text{ for all } v \in \mathbb{R}_+^N. \quad (4.10)$$

This is simply because Step 2.1 implies both $v \sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i)$ and $(\prod_{i \in N} v_i^{\frac{1}{n}}, \dots, \prod_{i \in N} v_i^{\frac{1}{n}})$

$$\sim (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} \prod_{i \in N} v_i^{\frac{1}{n}}) = (\underbrace{1, \dots, 1}_{n-1}, \prod_{i \in N} v_i).$$

Next, fix $v, w \in \mathbb{R}_+^N$. It is enough to show that $\left[\prod_{i \in N} v_i = \prod_{i \in N} w_i \right] \Rightarrow [v \sim w]$ and

$\left[\prod_{i \in N} v_i > \prod_{i \in N} w_i \right] \Rightarrow [v \succ w]$. If $\prod_{i \in N} v_i = \prod_{i \in N} w_i$, then (4.10) implies $v \sim (\prod_{i \in N} v_i^{\frac{1}{n}}, \dots, \prod_{i \in N} v_i^{\frac{1}{n}}) = (\prod_{i \in N} w_i^{\frac{1}{n}}, \dots, \prod_{i \in N} w_i^{\frac{1}{n}}) \sim w$. If $\prod_{i \in N} v_i > \prod_{i \in N} w_i$, then (4.10) implies $v \sim (\prod_{i \in N} v_i^{\frac{1}{n}}, \dots, \prod_{i \in N} v_i^{\frac{1}{n}}) \gg (\prod_{i \in N} w_i^{\frac{1}{n}}, \dots, \prod_{i \in N} w_i^{\frac{1}{n}}) \sim w$. Since \succsim is weakly monotonic, $(\prod_{i \in N} v_i^{\frac{1}{n}}, \dots, \prod_{i \in N} v_i^{\frac{1}{n}}) \succ (\prod_{i \in N} w_i^{\frac{1}{n}}, \dots, \prod_{i \in N} w_i^{\frac{1}{n}})$ and, by transitivity of \succsim , $v \succ w$. ■

5. References

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